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TRACTABLE FINITE APPROXIMATION OF CONTINUOUS NONCOOPERATIVE GAMES ON A PRODUCT OF LINEAR STRATEGY FUNCTIONAL SPACES

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Abstract

A method of the finite approximation of continuous noncooperative games is presented. The method is based on sampling the functional spaces, which serve as the sets of pure strategies of the players. The pure strategy is a linear function of time, in which the trend–defining coefficient is variable. The spaces of the players' pure strategies are sampled uniformly so that the resulting finite game is one whose payoff matrices are hypercubic. The presented method of finite approximation makes solutions tractable so that they can be easily implemented and practiced. The approximation procedure starts with alimited number of intervals, for which the respective finite game is built and solved. Then this number is gradually increased, and new, bigger, finite games are solved until an acceptable solution becomes sufficiently close to the same–type solutions at the preceding iterations. The closeness is expressed as the absolute difference between the trend-defining coefficients of the strategies from the neighbouring solutions. These distances should be decreasing once they are smoothed with respective polynomials of degree 2.

Key words: continuous game, linear strategy, payoff functional, finite approximation, acceptable solution.

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Introduction

Continuous noncooperative games are well-fitting mathematical models for describing and predicting interactions of subjects (players or persons) possessing continuums of their pure strategies [1, 2]. However, the continuity plays rather an obstructive role than a helpful one in envelopment. Even in the games with two players, finding and practicing a solution in mixed strategies is hardly tractable [3, 4]. In cases when a solution exists in pure strategies, it often is revealed not to be the only one. Thus, the problem of the single solution selection arises. Moreover, even if the solution is unique or can be selected as the single best one, it is not guaranteed to be simultaneously profitable and symmetric [2, 4, 5].

It is worth noting that the analytic solution search in continuous games is a very difficult task. As of 2020, no unified algorithmic theory of solving continuous noncooperative games has been built. Even in the games with two players, the algorithmization of the analytic solution search is possible only for special classes of payoff functions [2–4, 6]. Therefore, finite approximation of continuous noncooperative games is not just preferable but is also quite necessary [7, 8]. It should open a single way to interpret and advise the best actions for players, without losing information in the continuity.

Motivation

A special class of noncooperative games is one where the player's pure strategy is a time–varying function. This function is determined by a few parameters or coefficients which may vary through intervals. Therefore, the set of the player's pure strategies is a functional space. Such a game model is typical for social (behavioural) and economic interaction processes, where the player uses short–term time–varying strategies [3, 7, 7]. The player's pure strategy indicates the short–term dynamics of social and economic activity. Besides, linear short–term dynamics is typical for competitive development processes in ecosystems [9], industrial management [10], transportation [11, 12], energy accumulation [13], etc.

In the simplest but very common case, the strategy is a linear function of time. The time interval (not to be confused with intervals of the function coefficients) is usually pretty short because the dynamics of competitive activity is naturally volatile. Thus, each short time interval corresponds to some short-term trend of the competitive activity [3,4]. The whole process of the competitive activity is modelled as a series of such noncooperative games, where each game corresponds to its short term. Then, obviously, the games are required to be solved without delays.

The problems of a delayless solution and its uniqueness are addressed in studying approaches to finite approximations of continuous games. When the game is defined on finite–dimensional Euclidean subspaces, it can be approximated by appropriately sampling the sets of players' pure strategies [14,15], whereupon an approximating game is solved easily and relatively faster. The appropriateness herein means obtaining an approximating game whose content does not differ much from that of the initial, continuous, game. In the case when the game is defined on a product of functional spaces, a rigorous substantiation is required to sample the functional sets of players' pure strategies. As in the case of finite–dimensional Euclidean subspaces, this will allow appropriate sampling, without losing information in the continuity. Nevertheless, it is worth noting that an approximated solution (with respect to the initial game) can still be selected unilaterally in order to meet the demands and rules of the competitive system [3,4,16].

Goals and tasks to be fulfilled

Motivated by the above reasons, the goal is to develop a method of finite approximation of continuous noncooperative games whose kernels (the payoff functions of the players) are defined on a product of linear strategy functional spaces. For achieving the goal, the following tasks are to be carried out:

1. To formalize a continuous noncooperative game whose kernel (every player's payoff function) is defined on a product of linear strategy functional spaces. In such a game, the set of the player's pure strategies is a continuum of linear functions of time.
2. To rigorously substantiate a method of finite approximation. The selection of the single best game solution is to be justified.
3. To discuss the applicability and significance of the method. The contribution to the game theory field is to be clearly emphasized.

A continuous noncooperative game

In the above–mentioned continuous noncooperative game of N players, $N \in \mathbb{N}\backslash\{1\}$, each of the players uses short–term time–varying strategies determined by two coefficients. The short–term trend is defined by a real–valued coefficient which is multiplied by time t. The other coefficient is presumed to be known (i. e., it is a constant), and it is called an offset [3].

The pure strategy is valid on interval $[t_1; t_2]$ by $t_2 > t_1$, so the pure strategies of the player belong to a functional space of linear functions of time:

$$L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2].$$

Denote the trend–defining coefficient of the n–th player by b_n, where

$$b_n \in [b_n^{(min)}; b_n^{(max)}] \; by \; b_n^{(max)} > b_n^{(min)}. \tag{1}$$

If the n-th player's offset is a_n, then its set of pure strategies is

$$X_n = \{x_n(t) = a_n + b_n t, t \in [t_1; t_2] : b_n \in$$
$$\in [b_n^{(min)}; b_n^{(max)}] \subset \mathbb{R}\} \subset L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2], \quad n = \overline{1, N}. \tag{2}$$

The n–th player's payoff in situation $\{x_m(t)\}_{m=1}^N$ is $K_n(\{x_m(t)\}_{m=1}^N)$, $n = \overline{1, N}$. It is presumed that this payoff is an integral functional:

$$K_n(\{x_m(t)\}_{m=1}^N) = \int_{t_1}^{t_2} f_n(\{x_m(t)\}_{m=1}^N)dt, \tag{3}$$

where $f_n(\{x_m(t)\}_{m=1}^N)$ is an algebraic function of N functions $\{x_m(t)\}_{m=1}^N$ defined everywhere on $[t_1; t_2]$. Therefore, the continuous noncooperative game

$$\langle \{X_n\}_{n=1}^N, \{K_n(\{x_m(t)\}_{m=1}^N)\}_{n=1}^N \rangle \tag{4}$$

is defined on product

$$\bigtimes_{n=1}^N X_i \subset \bigtimes_{n=1}^N L[t_1; t_2] \subset \bigtimes_{n=1}^N \mathbb{L}_2[t_1; t_2] \tag{5}$$

of linear strategy functional spaces (2).

Acceptable solutions/situations

Since a series of games (4) on product (5) is to be solved for further usage in practice, the only acceptable solutions are equilibrium or/and efficient situations in pure strategies. Such situations are defined similarly to those in games on finite–dimensional Euclidean subspaces [1,2].

Definition 1. Situation $\{x_m^*(t)\}_{m=1}^N$ in game (4) on product (5) by conditions (1) – (3) is an equilibrium situation in pure strategies if inequalities

$$K_n(\{\{\{x_m^*(t)\}_{m=1}^N \backslash \{x_n^*(t)\}\} \cup x_n(t)\}) \leqslant$$
$$\leqslant K_n(\{x_m^*(t)\}_{m=1}^N) \quad \forall x_n(t) \in X_n \quad for \quad n = \overline{1, N} \tag{6}$$

are simultaneously true.

Definition 2. Situation $\{x_m^{**}(t)\}_{m=1}^N$ in game (4) on product (5) by conditions (1) – (3) is an efficient situation in pure strategies if inequalities

$$K_n(\{x^{**}\}_{m=1}^N) \leqslant K_n(\{x_m(t)\}_{m=1}^N)$$
$$by \quad \exists g \in \{\overline{1,N}\} \quad such\ that$$
$$K_g(\{x^{**}\}_{m=1}^N) < K_g(\{x_m(t)\}_{m=1}^N) \tag{7}$$

are impossible for any $x_n(t) \in X_n$ for $n = \overline{1,N}$.

The continuous noncooperative game can have the empty set of equilibria in pure strategies [2]. Moreover, every efficient situation can be too asymmetric, i. e. it is profitable for a subset of players and unacceptably unprofitable for another subset of players. In such cases, the game does not have an acceptable solution. Then the concepts of ε–equilibrium and ε–efficiency are useful (e. g., see [2,3]).

Definition 3. Situation $\{x_m^{*(\varepsilon)}(t)\}_{m=1}^N$ in game (4) on product (5) by conditions (1) – (3) is an ε–equilibrium situation in pure strategies for some $\varepsilon > 0$ if inequalities

$$K_n(\{\{\{x_m^{*(\varepsilon)}(t)\}_{m=1}^N \backslash \{x_n^{*(\varepsilon)}(t)\}\} \cup x_n(t)) \leqslant$$
$$\leqslant K_n(\{x_m^{*(\varepsilon)}(t)\}_{m=1}^N) + \varepsilon \quad \forall x_n(t) \in X_n$$
$$for \quad n = \overline{1,N} \tag{8}$$

are simultaneously true.

Definition 4. Situation $\{x_m^{**(\varepsilon)}(t)\}_{m=1}^N$ in game (4) on product (5) by conditions (1) – (3) is an ε–efficient situation in pure strategies for some $\varepsilon > 0$ if inequalities

$$K_n(\{x_m^{**(\varepsilon)}(t)\}_{m=1}^N) + \varepsilon \leqslant K_n(\{x_m(t)\}_{m=1}^N)$$
$$by \quad \exists g \in \{\overline{1,N}\} \quad such\ that$$
$$K_g(\{x_m^{**(\varepsilon)}(t)\}_{m=1}^N) + \varepsilon < K_g(\{x_m(t)\}_{m=1}^N) \tag{9}$$

are impossible for any $x_n(t) \in X_n$ for $n = \overline{1,N}$.

The equilibrium situations given by Definition 1 and Definition 3 (by admissible ε) and efficient situations by Definition 2 and Definition 4 (by admissible ε) are the acceptable solutions regardless of whether the game is finite or not. Obviously, the best consequence is when a situation is simultaneously equilibrium (by Definition 1) and efficient (by Definition 2). If such a situation does not exist, the most preferable is an efficient situation in which the sum of players' payoffs is maximal. However, if the payoffs are unacceptably asymmetric, then the best consequence is to find such admissible ε for which a situation is simultaneously equilibrium (by Definition 3) and efficient (by Definition 4). This approach relates to the method of solving games approximately by providing concessions (see [17]), where a payoff asymmetry is smoothed over by a compensation from the subset of players whose payoffs are unacceptably greater [3].

The finite approximation

It is obvious that, in game (4) on product (5) by conditions (1) – (3), the pure strategy of the player is determined by the trend–defining coefficient. Therefore, this game can be thought of as it is defined, instead of product (5) of linear strategy functional spaces (2), on N–dimensional hyperparallelepiped

$$\bigtimes_{n=1}^{N} [b_n^{(min)}; b_n^{(max)}] \subset \mathbb{R}^N. \tag{10}$$

This hyperparallelepiped is easily sampled by using a number of equal intervals along each dimension. Denote this number by S, $S \in \mathbb{N}\backslash\{1\}$. Then

$$B_n = \{b_n^{(min)} + (s-1) \cdot \frac{b_n^{(max)} - b_n^{(min)}}{S}\}_{s=1}^{S+1} =$$
$$= \{b_n^{(s)}\}_{s=1}^{S+1} \subset [b_n^{(min)}; b_n^{(max)}] \quad \forall n = \overline{1,N} \tag{11}$$

So, hyperparallelepiped (10) is substituted with grid $\bigtimes_{n=1}^N B_n$. Set (11) leads to a finite set

$$X_n^{(B_n)} = \{x_n(t) = a_n + b_n t, t \in [t_1; t_2] : b_n \in B_n \subset$$
$$\subset [b_n^{(min)}; b_n^{(max)}] \subset \mathbb{R}\} = \{x_{ns}(t) = a_n + b_n^{(s)} t\}_{s=1}^{S+1} \subset$$
$$\subset X_n \subset L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2] \tag{12}$$

of pure strategies (linear functions of time) of the n–th player, where

$$x_{n1}(t) = a_n + b_n^{(min)} t,$$
$$x_{n,S+1}(t) = a_n + b_n^{(max)} t.$$

Subsequently, game (4) on product (5) by conditions (1) – (3) is substituted with a finite game

$$\langle \{X_n^{(B_n)}\}_{n=1}^N, \{K_n(\{x_m(t)\}_{m=1}^N)\}_{n=1}^N \rangle$$
$$by \quad x_n(t) \in X_n^{(B_n)} \quad for \quad n = \overline{1,N} \tag{13}$$

defined on product

$$\bigtimes_{n=1}^{N} X_n^{(B_n)} \subset \bigtimes_{n=1}^{N} X_n \subset \bigtimes_{n=1}^{N} L[t_1; t_2] \subset \bigtimes_{n=1}^{N} \mathbb{L}_2[t_1; t_2] \tag{14}$$

of linear strategy functional subspaces (12). In fact, game (13) is a noncooperative $\bigtimes_{n=1}^N (S+1)$–game.

To perform an appropriate approximation via the sampling, number S is selected so that none of S_N sub–hyperparallelepipeds

$$\bigtimes_{n=1}^{N}[b_n^{(i_m)}; b_n^{(i_m+1)}] \subset \bigtimes_{n=1}^{N}[b_n^{(min)}; b_n^{(max)}] \subset \mathbb{R}^N$$

$$by \quad i_m = \overline{1, S} \quad and \quad m = \overline{1, N} \tag{15}$$

would contain significant specificities of payoff functionals (3) $\forall n = \overline{1, N}$. In fact, such specificities are extremals of these functionals.

Theorem 1. In game (4) on product (5) by conditions (1) – (3), the player's payoff functional achieves its maximal and minimal values on any closed sub–hyperparallelepiped of the trend–defining coefficients.

Proof. Every $f_n(\{x_m(t)\}_{m=1}^{N})$ $(n = \overline{1, N})$ is an algebraic function of linear functions $\{x_m(t)\}_{m=1}^{N}$ defined everywhere on $[t_1; t_2]$. Therefore, the integral in every functional (3) achieves some maximal and minimal values on any closed sub–hyperparallelepiped of the trend-defining coefficients.

Thus, Theorem 1 allows controlling extremals of payoff functionals (3) $\forall\ n = \overline{1, N}$ by the trend-defining coefficients. Moreover, Theorem 1 is easily expanded on closed sub–hyperparallelepipeds (15) for any number S. Consequently, if inequalities

$$\max_{\times_{n=1}^{N}[b_n^{(i_m)}; b_n^{(i_m+1)}]} K_j(\{x_m(t)\}_{m=1}^{N}) -$$

$$- \min_{\times_{n=1}^{N}[b_n^{(i_m)}; b_n^{(i_m+1)}]} K_j(\{x_m(t)\}_{m=1}^{N}) =$$

$$= \max_{\times_{n=1}^{N}[b_n^{(i_m)}; b_n^{(i_m+1)}]} \int_{t_1}^{t_2} f_j(\{x_m(t)\}_{m=1}^{N})dt -$$

$$- \min_{\times_{n=1}^{N}[b_n^{(i_m)}; b_n^{(i_m+1)}]} \int_{t_1}^{t_2} f_j(\{x_m(t)\}_{m=1}^{N})dt \leqslant \mu$$

$$\forall i_m = \overline{1, S} \quad and \quad \forall m = \overline{1, N} \quad and \quad \forall j = \overline{1, N} \tag{16}$$

are simultaneously true for some sufficiently small $\mu > 0$, then those μ–variations can be ignored. Thus, for the properly selected S and μ, game (4) on product (5) by conditions (1) – (3) can be approximated by finite game (13). The quality of the approximation can be comprehended by the following assertions.

Theorem 2. If $\{x_m^*(t)\}_{m=1}^{N}$ is an equilibrium in game (4) on product (5) by conditions (1) – (3), where functionals (3) $\forall n = \overline{1, N}$ are continuous, conditions (16) hold for some S and μ,

$$x_m^*(t) = a_m + b_m^* t \quad by \quad b_m^* \in [b_m^{(h_m)}; b_m^{(h_m+1)}]$$

$$for \quad h_m \in \{\overline{1, S}\} \quad and \quad \forall\ m = \overline{1, N}, \tag{17}$$

then every situation $\{x_m^{*(\varepsilon)}(t)\}_{m=1}^{N}$ for which

$$x_m^{*(\varepsilon)}(t) = a_m + b_m^{*(\varepsilon)} \quad by \quad b_m^{*(\varepsilon)} \in [b_m^{(h_m)}; b_m^{(h_m+1)}]$$

$$for \quad h_m \in \{\overline{1, S}\} \quad and \quad \forall m = \overline{1, N}, \tag{18}$$

is an ε–equilibrium for some $\varepsilon > 0$. As number S is increased, the value of ε can be made smaller.

Proof. Whichever integer S and the corresponding μ are, the value of ε always can be chosen such that inequalities (8) be true for every situation composed of strategies (18) by (17). It is obvious that, owing to the continuity of the functionals, the increasing number S allows decreasing the value of μ, which provides ε–equilibria to be closer to the equilibrium composed of strategies (17).

Theorem 3. If $\{x_m^{**}(t)\}_{m=1}^{N}$ is an efficient situation in game (4) on product (5) by conditions (1) – (3), where functionals (3) $\forall n = \overline{1, N}$ are continuous, conditions (16) hold for some S and μ,

$$x_m^{**}(t) = a_m + b_m^{**} t \quad by \quad b_m^{**} \in [b_m^{h_m}; b_m^{(h_m+1)}]$$

$$for \quad h_m \in \{\overline{1, S}\} \quad and \quad \forall m = \overline{1, N}, \tag{19}$$

then every situation $\{x_m^{**(\varepsilon)}(t)\}_{m=1}^{N}$ for which

$$x_m^{**(\varepsilon)}(t) = a_m + b_m^{**(\varepsilon)} \quad by \quad b_m^{**(\varepsilon)} \in [b_m^{(h_m)}; b_m^{(h_m+1)}]$$

$$for \quad h_m \in \{\overline{1, S}\} \quad and \quad \forall m = \overline{1, N}, \tag{20}$$

is an ε–efficient situation for some $\varepsilon > 0$. As number S is increased, the value of ε can be made smaller.

Proof. Whichever integer S and the corresponding μ are, value ε always can be chosen such that inequalities (9) be true for every situation composed of strategies (20) by (19). It is obvious that, owing to the continuity of the functionals, the increasing number S allows decreasing the value of μ, which provides ε–efficient situations to be closer to the efficient situation composed of strategies (19).

Hence, the finite approximation should start from some integer S, for which the respective finite $\times_{n=1}^{N}(S+1)$–game (13) is built and solved. Then this integer is gradually increased, and new, bigger, finite games are solved. The process can be stopped if the acceptable solution (whether it is an equilibrium, efficient, ε–equilibrium, or ε–efficient situation) by the last iteration does not differ much from the acceptable solutions (of the same type) by a few preceding iterations. Thus, if

$$\{x_n^{<l>*}(t)\}_{n=1}^{N} = \{a_n + b_n^{<l>*} t\}_{n=1}^{N} \in \bigtimes_{n=1}^{N} X_n^{(B_n)} \subset \bigtimes_{n=1}^{N} X_n \tag{21}$$

is an acceptable solution/situation at the l–th iteration, then the conditions of the sufficient closeness to the solutions at the preceding and succeeding iterations are as follows:

$$\sqrt{\int_{t_1}^{t_2}(x_n^{<l-1>*}(t) -_n^{<l>*}(t))^2\,dt} \geqslant$$

$$\geqslant \sqrt{\int_{t_1}^{t_2}(x_n^{<l>*}(t))^2\,dt} \quad \forall n = \overline{1,N} \qquad (22)$$

and

$$\max_{t\in[t_1;t_2]}|x_n^{<l-1>*}(t) - x_n^{<l>*}(t)| \geqslant$$

$$\geqslant \max_{t\in[t_1;t_2]}|x_n^{<l>*}(t) - x^{<l+1>*}(t)| \quad \forall n = \overline{1,N} \qquad (23)$$

by $l = 2, 3, 4, \ldots$

Theorem 4. Conditions (22) and (23) of the sufficient closeness for game (4) on product (5) by conditions (1) – (3) are expressed as trend–defining–coefficient closeness inequalities:

$$|b_m^{<l-1>*} - b_m^{<l>*}| \geqslant |b_m^{<l>*} - b_m^{<l+1>*}|$$

$$\forall n = \overline{1,N} \quad by \quad l = 2, 3, 4, \ldots \qquad (24)$$

Proof. Due to that

$$\sqrt{\int_{t_1}^{t_2}(x_n^{<l-1>*}(t) - x_n^{<l>*}(t))^2\,dt} =$$

$$= \sqrt{\int_{t_1}^{t_2}(a_n + b_n^{<l-1>*}t - a_n - b_n^{<l>*}\,t)^2\,dt} =$$

$$= \sqrt{\int_{t_1}^{t_2}(b_n^{<l-1>*} - b_n^{<l>*})^2\,t^2 dt} =$$

$$= \sqrt{(b_n^{<l-1>*} - b_n^{<l>*})^2\,(\frac{t_2^3}{3} - \frac{t_1^3}{3})} =$$

$$= |b_n^{<l-1>*} - b_n^{<l>*}|\sqrt{\frac{t_2^3 - t_1^3}{3}}$$

and

$$\max_{t\in[t_1;t_2]}|x_n^{<l-1>*}(t) - x^{<l>*}(t)| =$$

$$= \max_{t\in[t_1;t_2]}|(b_n^{<l-1>*} - b_n^{<l>*})t| = |b_n^{<l-1>*} - b_n^{<l>*}|t_2$$

(where time is presumed to be nonnegative), inequalities (22) and (23) are simplified explicitly:

$$|b_n^{<l-1>*} - b_n^{<l>*}|\sqrt{\frac{t_2^3 - t_1^3}{3}} \geqslant |b_n^{<l>*} - b_n^{<l+1>*}|\sqrt{\frac{t_2^3 - t_1^3}{3}} \quad \forall n = \overline{1,N}$$

and

$$|b_n^{<l-1>*} - b_n^{<l>*}|t_2 \geqslant |b_n^{<l>*} - b_n^{<l+1>*}|t_2 \quad \forall n = \overline{1,N},$$

whence they are expressed as just trend-defining-coefficient closeness inequalities (24).

If inequalities (24) hold for at least three iterations, the approximation procedure can be stopped. Clearly, the closeness strengthens if inequalities (24) strictly hold. However, inequalities (24) may not hold (i. e. at least one of them is violated) for a wide range of iterations, so it is better to require that polylines

$$\lambda_n(l) = |b_n^{<l>*} - b_n^{<l+1>*}| \quad \forall n = \overline{1,N} \quad by \quad l = 1, 2, 3, \ldots \qquad (25)$$

be decreasing on average. Herein, the term "on average" implies that, in the case when inequalities (24) do not hold (at least one of them is violated), polylines (25) are smoothed (approximated) with the respective polynomials of degree 2. The selection of the single best game solution relies on its convergence in accordance with polylines (25) being decreasing on average.

Selection of the single best game solution

Consider an example in which the selection of the single best game solution is justified. Let there be three players whose pure strategies are defined on $t \in [0; 90]$, and the sets of pure strategies of the first, second, and third player, are

$$X_1 = \{x_1(t) = 50 + b_1 t, t \in [0; 90] : b_1 \in$$

$$\in [-0.5; 0.5] \subset \mathbb{R}\} \subset L[0; 90] \subset \mathbb{L}_2[0; 90], \qquad (26)$$

$$X_2 = \{x_2(t) = 40 + b_2 t, t \in [0; 90] : b_2 \in$$

$$\in [-0.6; 0.6] \subset \mathbb{R}\} \subset L[0; 90] \subset \mathbb{L}_2[0; 90], \qquad (27)$$

$$X_3 = \{x_3(t) = 80 + b_3 t, t \in [0; 90] : b_3 \in$$

$$\in [-0.3; 0.3] \subset \mathbb{R}\} \subset L[0; 90] \subset \mathbb{L}_2[0; 90], \qquad (28)$$

respectively (Figure 1). Their respective payoff functionals are

$$K_1(x_1(t), x_2(t), x_3(t)) =$$

$$\int_0^{90}(-2x_1^2 - 3x_1x_2 + x_3^2 + x_1x_2x_3 - \frac{x_1}{4x_2 + x_3 + 60})dt, \qquad (29)$$

$$K_2(x_1(t), x_2(t), x_3(t)) =$$

$$= \int_0^{90} 22 \cdot (3x_1x_3 - 8x_2 - \frac{x_1}{x_3^2})dt, \qquad (30)$$

$$K_3(x_1(t), x_2(t), x_3(t)) = \int_0^{90}\frac{256x_1x_2x_3}{x_1 + x_2 + x_3 + 20}dt. \qquad (31)$$

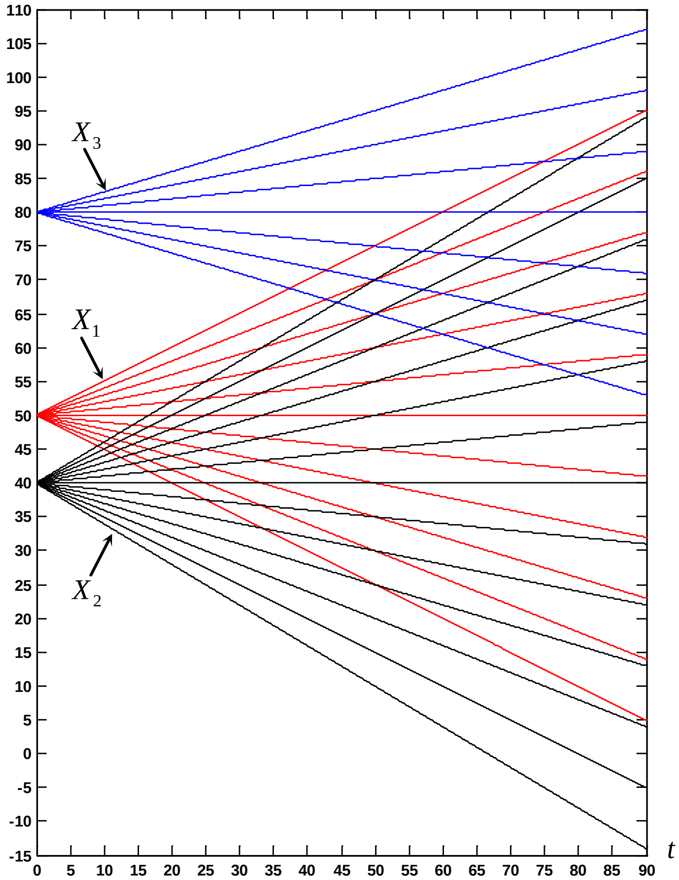

Fig. 1: The sets (functional spaces) of the players' pure strategies (26) – (28)

Consequently, this game can be thought of as it is defined on parallelepiped

$$[-0.5; 0.5] \times [-0.6; 0.6] \times [-0.3; 0.3] \subset \mathbb{R}^3. \quad (32)$$

It is easy to see that each of functionals (29) – (31) is continuous. Therefore, Theorem 2 and Theorem 3 ensure a possibility of the finite approximation. For making this, the number of intervals is gradually increased from 5 up to 30, i. e. $S = \overline{5, 30}$ and thus $6 \times 6 \times 6$–game, $7 \times 7 \times 7$–game, ..., $31 \times 31 \times 31$–game are solved. Each of these games has a single equilibrium situation and $S + 1$ efficient situations (Figure 2). Every situation has $b_3^{<l>*} = 0.3$ ($l = \overline{1, 26}$), so the third player has its best strategy $x_3(t) = 80 + 0.3t$ in an acceptable solution. In every equilibrium situation, the trend–defining coefficient of the second player is -0.6, whereas the trend–defining coefficient of the first player is 0.5 in every efficient situation. Furthermore, every situation with strategies $x_1(t) = 50 + 0.5t$ and $x_3(t) = 80 + 0.3t$ is efficient (Figure 3). In other words, every situation

$$\{x_1(t), x_2(t), x_3(t)\} = \{50 + 0.5t, 40 + b_2^{(s)} t, 80 + 0.3t\} =$$

$$= \{50 + 0.5t, 40 + ((s-1) \cdot \frac{1.2}{S} - 0.6)t, 80 + 0.3t\}$$

$$\forall s = \overline{1, S+1} \quad (33)$$

is efficient. However, situations (33) are not equilibrium except for situation

$$\{x_1(t), x_2(t), x_3(t)\} = \{50 + 0.5t, 40 - 0.6t, 80 + 0.3t\} \quad (34)$$

in the respective $6 \times 6 \times 6$–game (at the very first iteration, with $S = 5$; the single circled dot is seen in Figure 2 and, iteration–wise, in Figure 3).

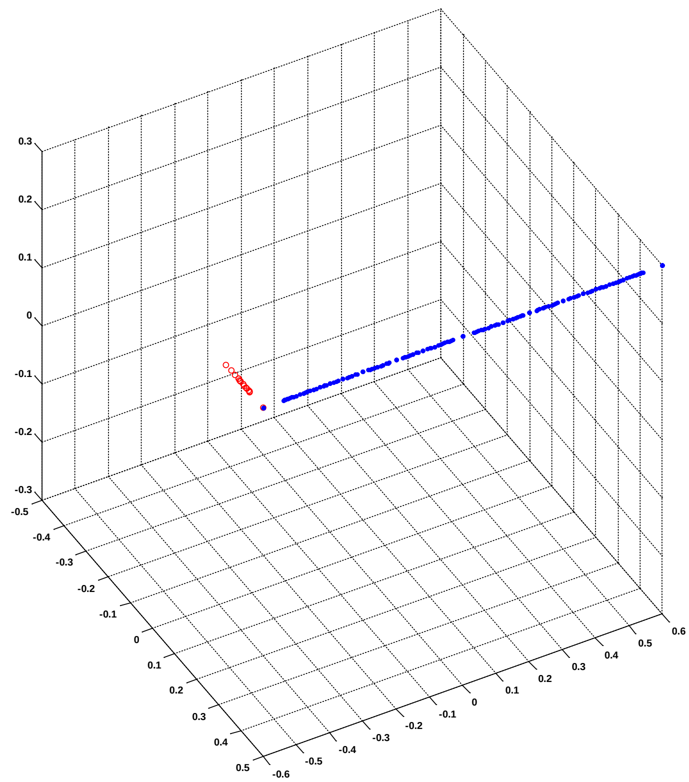

Fig. 2: All 26 equilibria (circle–marked) and 481 efficient situations (dot–marked) visualized on parallelepiped (32) in the solved trimatrix games

Figure 3 also shows that the single equilibrium is not "stable" as the iterations progress. Nevertheless, a very important fact here is that the maximal sum of players' payoffs in equilibrium situations is equal to the minimal sum of players' payoffs in efficient situations. This parity exists in equilibrium–and–efficient situation (34). Consequently, the sum of players' payoffs in other efficient situations is greater than that in equilibrium situations. Moreover, the average payoff of every player in efficient situations is greater than that in equilibrium situations. This implies that the players will prefer the efficient situations to the equilibria. The maximal sum of players' payoffs in efficient situations is achieved in situation

$$\{x_1^{**}(t), x_2^{**}(t), x_3^{**}(t)\} =$$

$$= \{50 + 0.5t, 40 + 0.6t, 80 + 0.3t\}. \quad (35)$$

Therefore, situation (35) is the best acceptable solution for the players.

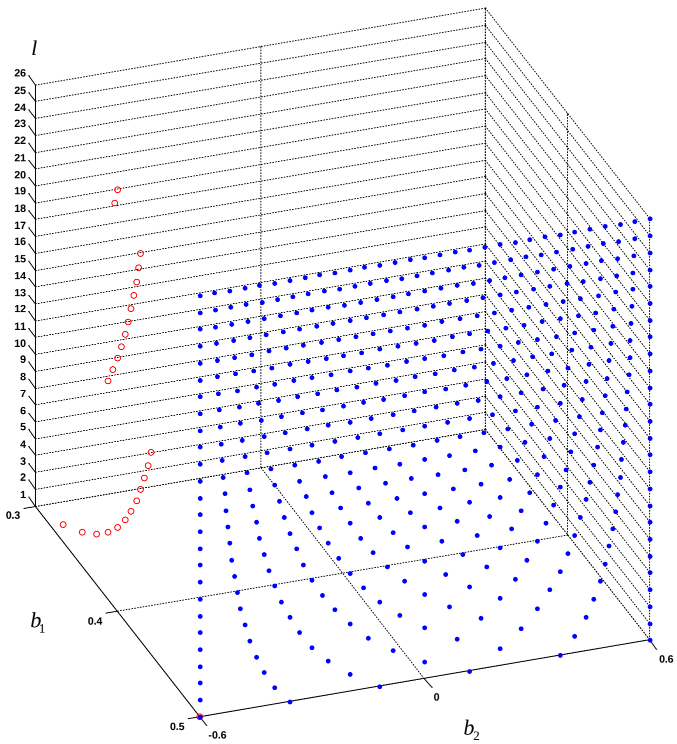

Fig. 3: The evolution of the single equilibrium and $S+1$ efficient situations through 26 iterations

It is worth noting that inequalities (24) in this case trivially hold. Then, obviously, polylines (25) turning into zeros are needless. However, this is probably just a "suitable" example. In fact, in "catching" the single best game solution, the maximization of the sum of players' payoffs is as influential as finding polylines (25) decreasing on average. This is a sufficient mindset to start searching for the solution. This approach is kind of processing the game from the simplest treatment to building it more complicated unless the framework "stabilizes".

Discussion

Short–term strategies can always be approximated as linear functions of time. The presented method of finite approximation makes solutions tractable so that they can be easily implemented and practiced. So, the finite approximation method specifies and establishes the applicability of continuous noncooperative games on a product of linear strategy functional spaces. Mainly, it concerns modelling social (behavioural) and economic interaction processes, where the player can use a continuum of short–term time–varying strategies. Other processes, in which short–term strategies can be linearized for the simpler consideration, are intrinsic competitions in ecosystems [9], health–and–safety engineering [18], managing traffic and storage regulations [11–13], etc.

However, the justification of the single best game solution referred to as an acceptable situation still cannot be algorithmized in general. It requires considering the sum of players' payoffs and convergences, where the payoffs should "stabilize" and the acceptable situations are expected to cluster without bouncing.

In modelling, any process with continuities is tried to be discretized into short–term parts. Therefore, the presented finite approximation method is quite significant. It allows avoiding too complicated solutions resulting from game continuities and, moreover, functional spaces of pure strategies. Such a "deeinstellungization" [19] of the continuous noncooperative games is a promising approach to effectively distribute limited resources under uncertainty or/and growing demands [2, 3], and remove negative impact of sophisticated mindset on making optimal/tractable decisions as well [20].

Conclusion

For solving continuous noncooperative games on a product of linear strategy functional spaces, a method of their finite approximation is presented, which is based on sampling the linear strategy functional spaces. The sets (i. e. the spaces) of the players' pure strategies are sampled uniformly so that the resulting finite game is defined on a multidimensional cube. The respective payoff matrices are hypercubic. The approximation procedure starts with a limited number of intervals, for which the respective finite game is built and solved. Then this number is gradually increased (the increment is not defined for a general case), and new, bigger, finite games are solved until an acceptable solution becomes sufficiently close to the same–type solutions at the preceding iterations. The closeness is expressed in terms of the respective functional spaces, which is simplified by Theorem 4, giving just the absolute difference between the trend–defining coefficients of the strategies from the neighbouring solutions. These distances should be decreasing once they are smoothed with respective polynomials of degree 2.

The presented finite approximation method is a contribution to the game theory field encompassed with noncooperative games whose players' strategies are linear functions of time. The contribution also consists in solving the problem of solution uniqueness. It allows effectively modelling social (behavioural), economic, ecological, managerial, health–and–safety engineering and industrial interaction processes, where players (any number of them) use short–term linear strategies corresponding to short–term trends of their activity. The practical effectiveness is ensured by that solving a series of such short–term noncooperative games is faster owing to the whole process is discretized and thus much simplified. A question of the game finite approximation for cases of nonlinear strategy spaces (when, say, the player's strategy space is of parabolas or cubic polynomials) is believed to be answered in the similar manner, although some peculiarities concerning the continuity of the payoff functionals may weaken the impact of Theorem 2 and Theorem 3.

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